ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, Nos. 2-3, 1978

MINKOWSKI SPACES WITH EXTREMAL DISTANCE FROM THE EUCLIDEAN SPACE

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ABSTRACT

It is proved that if the Banach–Mazur distance between an n-dimensional Minkowski space B and l_2^n satisfies $d(B, l_2^n) \geqq c\sqrt{n}$ (for some constant $c > 0$ and for big n) then B contains an $A(c)$-isomorphic copy of l_1^k (for $k \sim \log\log\log n$). In the special case $d(B, l_2^n) = \sqrt{n}$, B contains an isometric copy of l_1^k for $k \sim \log n$.

Introduction

We recall that a Minkowski space is a finite dimensional, normed linear space (i.e. a finite dimensional Banach space). The Banach–Mazur distance between two n-dimensional Minkowski spaces B_1 and B_2 is defined as $d = d(B_1, B_2) = \inf_{T: B_1 \to B_2} \|T\| \, \|T^{-1}\|$, where the infimum is taken over all isomorphisms from B_1 onto B_2. Actually $\rho = \log d$ is a metric on the space of n-dimensional Minkowski spaces but it is more convenient to use d. Clearly $d(B_1, B_2) \geqq 1$ and $d(B_1, B_2) = 1$ if and only if B_1 and B_2 are isometric Minkowski spaces. If $d(B_1, B_2) \leqq 1 + \varepsilon$ we will say that B_1 and B_2 are ε-isometric spaces. F. John [11] proved, that the distance from any n-dimensional Banach space B to l_2^n is $d(B, l_2^n) \leqq \sqrt{n}$. The maximal distance is attained e.g. for two classical spaces l_1^n, l_∞^n: $d(l_1^n, l_2^n) = d(l_\infty^n, l_2^n) = \sqrt{n}$ (see [8]). In this paper we will prove two main theorems:

THEOREM 1. *For every positive integer k there is a positive integer n, such that every n dimensional Banach space B, satisfying $d(B, l_2^n) = \sqrt{n}$, contains a k-dimensional subspace $E_0 \subset B$, which is isometric to l_1^k. Asymptotically (when $n \uparrow \infty$) we have the relation $k(n) \geqq [1/(2\ln 12)]\ln n$.*

Obviously this estimate is exact (up to the coefficient of $\ln n$) since l_∞^n contains an l_1^k with k not greater than $\log_2 n$.

Received April 10, 1977

THEOREM 2. *For any positive integer k and for any positive constant $c>0$ there is a positive integer $n=n(k,c)$ and a positive constant $c_1=c_1(c)$, such that every n-dimensional Banach space B satisfying $d(B,l_2^n)\geqq c\sqrt{n}$ contains a k-dimensional subspace $E\subset B$, such that $d(E,l_1^k)\leqq c_1$. Asymptotically $k(n,c)$ $(n\uparrow\infty)$ is bounded from below by the function* $\log\log\log n$.

REMARK. Theorem 2 is correct also when instead of the constant $c>0$ we take a slowly decreasing function $c(n)\to 0$ $(n\to\infty)$. Namely if $c(n)=(\log\log\log n)^{-o(1)}$ then, by Theorem 2, for every positive integer k and for every $\varepsilon>0$ there exists $n(k,\varepsilon,c(n))$ such that every n-dimensional space B, satisfying $\dim B=n\geqq n(k,\varepsilon,c(n))$ and $d(B,l_2^n)\geqq c(n)\sqrt{n}$, contains a k-dimensional subspace ε-isometric to l_1^k. We prove this result at the end of Appendix 2.

The following result is a trivial consequence of Theorem 2.

COROLLARY 1. *Let X be a super-reflexive infinite dimensional Banach space (for definition of super-reflexivity see* [9]). *Define $d_X(n)=\sup_{\dim E=n}^{E\subset X}\{d(E,l_2^n)\}$, where the supremum is taken over all n-dimensional subspaces of X. Then $d_X(n)=o(\sqrt{n})$.*

If we do not take into account the problem of the estimate on the dimension of the subspace isomorphic to l_1^k, then Theorem 2 claims only that an $n$dimensional Banach space B satisfying $d(B,l_2^n)\geqq c\sqrt{n}$ is of type[†] not better than 1 (asymptotically when $n\to\infty$). In this connection it is interesting to notice that the following result was obtained in [6]: if an n dimensional space B is of type p and of cotype q, then $d(B,l_2^n)\leqq cn^{2(1/p-1/q)}$ (c depends on the type and cotype constants of B). It is easy to see that there is a gap between these two results and that Theorem 2 gives an essentially stronger result (but only in the case when the distance is close to $\sqrt{n}$).

This article has two more contact points with the results in [6]. From the results of [6], §2, it is easily seen, that if $d(B,l_2^n)\leqq o(\sqrt{n})$, then B contains a k-dimensional ($k\geqq\log n$) subspace E, which is ε-isometric to l_2^k, and there is a projection $P:B\to E$ such that $\|P\|\leqq o(\sqrt{k})$. Besides if B does not contain subspaces ε-isometric to l_1^m (when $m\to\infty$), then $k\geqq n^\alpha$ for some $\alpha>0$. This with Theorem 2 implies that if B contains no subspaces ε-isometric to l_1^m (for some $m\to\infty$ when $n\to\infty$) then there is a subspace $E\subset B$ which is ε-isometric to l_2^k for $k\geqq n^\alpha$ (for some fixed $\alpha>0$) and there is a projection $P:B\to E$ with $\|P\|\leqq o(\sqrt{k})$.

This result (without the estimate on k) was obtained by different methods (by using the results of [1], [17]) by W. J. Davis and W. B. Johnson [3].

[†] For a detailed discussion of type and cotype see [17].

Let us state a related open question: let E be a k-dimensional subspace of B ($\dim B = n$), $d(E, l_2^k) \leqq 2$ and let the best projection $P: B \to E$ have norm $\|P\| \approx c\sqrt{k}$. Is it true that B contains a subspace 2-isomorphic to l_1^m ($m \to \infty$ when $n \to \infty$) ?

The following question is another contact point with [6]: is it true that if the conditions of Theorem 2 are satisfied, then B contains a subspace isomorphic to l_∞^m (for some $m \to \infty$ when $n \to \infty$), or B contains a complemented subspace isomorphic to l_1^k ($k \to \infty$ when $n \to \infty$)? From the results of [6] it is easily seen that if the unit ball of B ($\dim B = n$) has $f(n)$ $(n-1)$-dimensional faces, then the dimension k of a subspace contained in B and 2-isomorphic to l_2^k is not bigger than $c \ln f(n)$. Hence, if $\ln \ln f(n)/\ln n \to 0$, then B cannot be of finite cotype (by [6]) and, by a theorem of Maurey and Pisier [17], B contains a subspace isomorphic to l_∞^m with $m \to \infty$ when $n \to \infty$. By duality we get that if $\phi(n)$ is the number of extreme points of the unit ball of an n-dimensional space B and $(\ln \ln \phi(n))/\ln n \to 0$, then B contains a uniformly (when $n \to \infty$) complemented subspace, isomorphic to l_1^k ($k \to \infty$ when $n \to \infty$). It is not difficult to check that in this case $d(B, l_2^n) \geqq n^{\frac{1}{2}-o(1)}$. (This observation is due to J. Lindenstrauss.)

B. Maurey [16] has shown that if $(\ln \phi(n))/n \to 0$, then B contains a subspace 2-isomorphic to l_1^k with $k \to \infty$ when $n \to \infty$. But in this case nothing is known about the relative projection constants of subspaces isomorphic to l_1^k.

Let us state another important problem, which we formulate rather vaguely.

Let B be an n-dimensional Minkowski space satisfying $d(B, l_2^n) = d < \sqrt{n}$. When is there no other space B' satisfying $d(B', l_2^n) = d(B, B')d(B, l_2^n)$ and $d(B', l_2^n) > d$?

If there is no such space B' then Theorem 2 would follow trivially from Theorem 1 and the estimate could be improved.

The proofs of Theorems 1 and 2 are based on two variational results: a theorem of F. John ([11]) and a lemma of Larman and Mani (see [14]), which is close in its spirit to the Dvoretzky–Rogers lemma ([4]). We quote these results, but the lemma of Larman and Mani is sufficient only for the proof of Theorem 1. In order to prove Theorem 2 we need an additional result which is an isomorphic variant of the lemma of Larman and Mani. Our first proof of this generalization was technically complicated. But afterwards B. Maurey gave an elegant proof of a stronger result, which we will use here.

LEMMA 0.1 (Maurey). *Let E and B be Minkowski spaces,* $\dim E = n$, $d(E, l_2^n) = d$, $E \subset B$ *and* $\dim B/E = k$. *Then there is an absolute constant* $b > 0$ *such that*

$$d(B, l_2^{n+k}) \leqq 3d + 2^{bk}.$$

The proof of Maurey's lemma is given in Appendix 1.

Another technical result, needed in the proof of Theorem 2, is the following:

THEOREM 3. *Let $L_\infty(T,\mu)$ be the space of essentially bounded functions on a set T with a probability measure μ ($\mu(T)=1$) and let $A<\infty$ be a constant. Suppose there are m functions $\{f_1(t),\cdots,f_m(t)\}\subset L_\infty(T,\mu)$ with $\|f_i\|_{L_\infty}\leqq A$, $1\leqq i\leqq m$ which are orthonormal in $L_2(T,\mu)$. Then there are k functions ($k\sim\log\log m$) $f_{i_1}(t),\cdots,f_{i_k}(t)$ such that* span$\{f_{i_1},\cdots,f_{i_k}\}$ *in $L_\infty(T,\mu)$ norm is c_1-isomorphic to l_1^k, where c_1 depends only on A.*

Our proof of Theorem 3 is complicated. W. B. Johnson showed us a simplification of the proof which uses results of A. Brunel and L. Sucheston [1], based on the combinatorial theorem of Ramsey. His proof gives, however, no estimate of $k(m)$. Therefore, in the main text we give the simple proof of Johnson and in Appendix 2 we show a sketch of our first proof of the theorem.

We thank J. Lindenstrauss, for his active interest in our work resulting in the above mentioned (belonging to B. Maurey and W. B. Johnson) simplifications of the paper.

Proof of the main results

1. Our problem is to find an isometric (isomorphic) copy of l_1^k in every n-dimensional Banach space B, which satisfies the conditions of Theorem 1 (or 2). For this purpose it is enough to find unit vectors $\{\eta_1,\cdots,\eta_k\}\subset S(B)$, where $S(B)$ is the surface of the unit ball of B, such that for any k-tuple of scalars $\{a_j\}_{j=1}^k$,

$$\left\|\sum_{j=1}^k a_j\eta_j\right\| = \sum_{j=1}^k |a_j|$$

$$\left(c_1\sum_{j=1}^k |a_j| \leqq \left\|\sum_{j=1}^k a_j\eta_j\right\| \leqq c_2\sum_{j=1}^k |a_j|\right). \tag{1.1}$$

Observe that if for every vector of signs $\varepsilon=\{\varepsilon_j=\pm 1\}_{j=1}^k$, there exists a functional f_ε, such that $\|f_\varepsilon\|=1$ and $f_\varepsilon(\eta_j)=\varepsilon_j$ ($1\leqq j\leqq k$), then for every k-tuple of scalars $\{a_j\}_{j=1}^k$ satisfying $\varepsilon_j a_j \geqq 0$,

$$\sum_{j=1}^k |a_j| = f_\varepsilon\left(\sum_{j=1}^k a_j\eta_j\right) \leqq \left\|\sum_{j=1}^k a_j\eta_j\right\| \leqq \sum_{j=1}^k |a_j|.$$

Thus, if for some set of unit vectors $\{\eta_j\}_{j=1}^k$ and for every choice of signs $\varepsilon = (\varepsilon_j)_{j=1}^k$, there exists a functional f_ε such that $\| f_\varepsilon \| = 1$ and $\{f_\varepsilon(\eta_j) = \varepsilon_j\}_{j=1}^k$, then $\operatorname{span}\{\eta_j\}_{j=1}^k$ is isometric to l_1^k. (Similarly, if for every choice of signs $\varepsilon = (\varepsilon_j)_{j=1}^k$ there exists a functional f_ε, such that $\| f_\varepsilon \| < c_1$ and $\{f_\varepsilon(\eta_j)\varepsilon_j \geqq c_2 > 0\}_{j=1}^k$, then $\operatorname{span}\{\eta_j\}_{j=1}^k$ is c_1/c_2 isomorphic to l_1^k.)

So, our task is to find two such sets $\{\eta_j\}_{j=1}^k$ and $\{f_\varepsilon \mid \varepsilon = (\varepsilon_j = \pm 1)_{j=1}^k\}$. For this purpose we use two variational propositions. One of them is due to F. John ([11], the way the result is used here is discussed in [12]).

PROPOSITION I.1. *Let E be the ellipsoid of minimal volume containing the unit ball of B. We can consider l_2^n as the Euclidean space generated by the unit ball E. Then there exists a finite set of vectors $\{y_j\}_{j=1}^s$ in $S(B)$ and positive scalars $\{\lambda_j\}_{j=1}^s$ $(n \leqq s \leqq n(n+1)/2)$ satisfying:*

(a) $\| y_j \|_B = \| y_j \|_{l_2^n} = 1$,

(b) $\sum_{j=1}^s \lambda_j = n;\ \lambda_j > 0 \quad (1 \leqq j \leqq s)$,

(c) $\forall x \in B,\ x = \sum_{j=1}^s \lambda_j (x, y_j) y_j$,

(d) $\| x \|_{l_2} \leqq \| x \|_B \leqq \sqrt{n} \| x \|_{l_2}$

((d) *is a consequence of* (a), (b) *and* (c)).

Substituting $x = y_j$ in (c) we can get $\lambda_j \leqq 1$ $(1 \leqq j \leqq s)$.

PROOF OF THEOREM 1. Let $x_0 \in S(B)$ be a vector, where the minimum of the l_2^n norm is attained. $d(B, l_2^n) = \sqrt{n}$ implies $\| x_0 \|_{l_2^n} = 1/\sqrt{n}$. From Proposition I.1(c) we get:

$$
\begin{aligned}
1 = \| x_0 \|_B &= \left\| \sum_{j=1}^s \lambda_j (x_0, y_j) y_j \right\|_B \leqq \sum_{j=1}^s \lambda_j \left| (x_0, y_j) \right| \leqq \left(\sum_{j=1}^s \lambda_j \right)^{1/2} \left(\sum_{j=1}^s \lambda_j (x_0, y_j)^2 \right)^{1/2} \\
&= \sqrt{n} \| x_0 \|_{l_2^n} = 1.
\end{aligned} \tag{1.2}
$$

Hence, there is a constant $c > 0$ satisfying $\lambda_j^{1/2} = c \lambda_j^{1/2} \left| (x_0, y_j) \right|$ $(1 \leqq j \leqq s)$. Since $1/n = \| x_0 \|_{l_2^n}^2 = \sum_{j=1}^s \lambda_j (x_0, y_j)^2$, we get $\left| (x_0, y_j) \right| = 1/n$ $(1 \leqq j \leqq s)$.

The sign of y_j $(1 \leqq j \leqq s)$ in Proposition I.1(c) is insignificant, so we can choose $\{y_j\}_{j=1}^s$ to satisfy $(x_0, y_j) = 1/n$. Now, let x be any vector satisfying $\| x \|_B = 1$ and $\| x \|_{l_2^n} = 1/\sqrt{n}$. Since $\left| (x, y_j) \right| = 1/n$ $(1 \leqq j \leqq s)$, there is a vector of signs $\varepsilon = (\varepsilon_j = \pm 1)_{j=1}^s$ satisfying $n(x, y_j) = \varepsilon_j$ $(1 \leqq j \leqq s)$. Let $B^* \ni f(\| f \|_{B^*} = 1)$ be a supporting functional to $S(B)$ in x. Then f is also a supporting functional to $(1/\sqrt{n})E$, since in x the minimum of l_2^n norm on $S(B)$ is attained, and hence $f = nx$. Every functional in the set $\{f \mid f = nx, x \in M\}$, where $M = \{x \mid \| x \|_{B_n} = 1;$ $\| x \|_{l_2^n} = 1/\sqrt{n}\}$, defines a vector of signs $\{f(y_j)\}$. In order to investigate the properties of the set M we need a result of Larman and Mani ([14]).

PROPOSITION I.2. *Let B be an n-dimensional Banach space such that $d(B, l_2^n) = d$. Suppose that the unit ball of l_2^n is the Euclidean ball on which the Banach–Mazur distance is attained and $\| x \|_B \geqq \| x \|_{l_2^n} \geqq (1/d)\| x \|_B$. For any $\delta > 0$ there is a positive integer $m = ([d/4], [\delta^2 d^2/4])$ such that there exist at least m unit vectors $\{x_i\}_{i=1}^m$ and m orthogonal vectors $\{e_i\}_{i=1}^m$ satisfying*:

(i) $\| x_i \|_B = 1, \quad \| x_i \|_{l_2^n} = 1/d = \| e_i \|_{l_2^n}$,

(ii) $\| e_i - x_i \|_{l_2^n} \leqq \delta \| x_i \|_{l_2^n}$.

In the sequel we use this proposition with $d = n^{1/2}$ and $\delta = n^{1/4}$. Hence, $m = \frac{1}{4} n^{1/2}$.

By Propositions I.1 and I.2 we get a set of unit vectors (in B) $\{y_j\}_{j=1}^s$ and a set of normalized functionals (in B^*) $\{nx_i\}_{i=1}^m$, such that the matrix $T = \{n(x_i, y_j)\}_{j=1,\cdots,s}^{i=1,\cdots,m}$ has ± 1 as its entries.

It is more convenient to construct the subspace isometric to l_1^k in B^*. As $d(B, l_2^n) = d(B^*, l_2^n)$ the result obtained for B^* and the estimate on $k(n)$ hold for B too. We consider the B^* norm as the conjugate norm of B induced by the inner product (x, y), which is determined by the ellipsoid E. Hence in the same n-dimensional linear space we have defined three norms: the B-norm, the l_2^n-norm and the conjugate B^*-norm.

We show that there is a k-tuple of vectors $\{e_1, \cdots, e_k\}$ in B^*, $\| e_i \|_{B^*} = 1$, which is isometrically equivalent to the standard basis of l_1^k. This k-tuple is chosen as a subset of $\{nx_i\}_{i=1}^m$. The isometry is proved as it is explained at the beginning of this section and the functionals over B^*, $\{f_\varepsilon\}$ are chosen as a subset of the $\{y_j\}_{j=1}^s$.

Therefore we have to find a set of k indices $(i_1, \cdots, i_k)$ such that the set of sign vectors $\{(n(x_{i_p}, y_j))_{p=1}^k\}_{j=1}^s$ will contain all the possible 2^k vectors of signs. Denote by U_k the $k \times 2^k$-dimensional matrix having all the possible 2^k vectors of signs as its columns. We want to find in the matrix T those rows and columns which constitute the matrix U_k.

Let R^s be the s-dimensional vector space with the scalar product $\langle a, b \rangle = \sum_{j=1}^s \lambda_j a_j b_j (a = (a_1, \cdots, a_s);\ b = (b_1, \cdots, b_s))$. By Proposition I.1, for any $z_1, z_2 \in l_2^n$ $(z_1, z_2) = \sum_{j=1}^s \lambda_j (z_1, y_j)(y_j, z_2) = \langle \hat{z}_1, \hat{z}_2 \rangle$ where $\hat{z}_1 = ((z_1, y_j)_{j=1}^s)$, $\hat{z}_2 = ((z_2, y_j)_{j=1}^s) \in R^s$. Thus the scalar product in R^s of two rows i_1, i_2 in the matrix T is equal to (nx_{i_1}, nx_{i_2}). By Proposition I.2, these vectors are "almost orthogonal". This will help us to solve the combinatorial problem of finding the indices $i_1, \cdots, i_k$.

LEMMA I.3. *Let $\{e_i\}_{i=1}^m$ $(m \leqq n)$ be an orthogonal system in l_2^n and let f be a linear functional. Then there are no more than $t \leqq 1/\varepsilon^2$ vectors from $\{e_i\}_{i=1}^m$ satisfying*

$$|f(e_i)| \geqq \varepsilon \|f\|_{l_2^n} \|e_i\|_{l_2^n}.$$

PROOF. Suppose $\|e_i\|_{l_2^n} = 1$. It is clear that $\|f\|^2_{l_2^n} \geqq \sum_{i=1}^m |e_i(f)|^2$. If $|f(e_i)| \geqq \varepsilon \|f\|_{l_2^n}$ for more than t vectors e_i, we will get $\|f\|^2_{l_2^n} > 1/\varepsilon^2 \cdot \varepsilon^2 \|f\|^2_{l_2^n}$, which is a contradiction.

LEMMA I.4. *Let $\{x_i\}$ be the vectors from Proposition* I.2. *Then for every $f \in l_2^n$*

$$|f(x_i)| \leqq \|f\|_{l_2} \|x_i\|_{l_2} (\varepsilon + \delta)$$

except for at most $[1/\varepsilon^2]$ *vectors* $\{x_i\}$.

PROOF. By Proposition I.2 and Lemma I.3, except for at most $[1/\varepsilon^2]$ vectors $\{x_i\}$, we get

$$|f(x_i)| = |f(x_i - e_i) + f(e_i)| \leqq \|f\|_{l_2} \|x_i - e_i\|_{l_2} + |f(e_i)|$$

$$\leqq \|f\|_{l_2} \|x_i\|_{l_2} \delta + \|f\|_{l_2} \|e_i\|_{l_2} \varepsilon = \|f\|_{l_2} \|x_i\|_{l_2} (\varepsilon + \delta). \qquad \text{Q.E.D.}$$

Let $\{\xi_i = (0, \cdots, 0, 1_i, 0, \cdots, 0)\}_{i=1}^s$ be the standard basis of R^s. For a set of indices $I \subset [1, \cdots, s]$ define the functional $f_I \in (R^s)^*$ by

$$f_I(\xi_i) = \begin{cases} 0, & i \notin I \\ 1, & i \in I. \end{cases}$$

Thus $f_I(\sum_{i=1}^s a_i \xi_i) = \sum_{i \in I} \lambda_i a_i$. Then, by definition of the scalar product in R^s, $\|f_I\|_{l_2} = \sqrt{\sum_{i \in I} \lambda_i}$.

By Lemma I.4 for every ε, δ all the vectors $\{x_i\}$ (except $[1/\varepsilon^2]$ of them) satisfy:

$$|f_I(\hat{x}_i)| = \left| \sum_{j \in I} \lambda_j (x_i, y_j) \right| \leqq (\delta + \varepsilon) \|f_I\|_{l_2} \|x_i\|_{l_2} = (\delta + \varepsilon) \left(\frac{\sum_{i \in I} \lambda_i}{n} \right)^{1/2}.$$

Setting $\mu(I) = (\sum_{i \in I} \lambda_i)/n$ (the relative λ-measure of the set I) we get:

$$(1.3) \qquad |f_I(\hat{x}_i)| \leqq (\delta + \varepsilon) \sqrt{\mu(I)}$$

for all the vectors x_i, except $[1/\varepsilon^2]$ of them. By choosing $1/n^{1/4} = \delta < \varepsilon = \frac{1}{6}\sqrt{\mu(I)}$, we get

$$(1.4) \qquad \left| \sum_{j \in I} \lambda_i (x_i, y_j) \right| = |f_I(\hat{x}_i)| \leqq \tfrac{1}{3}\mu(I)$$

for set I such that $\mu(I) > 36/n^{1/2}$.

On the other hand:

$$(1.5) \qquad \sum_{j \in I} \lambda_j |(x_i, y_j)| = \frac{1}{n} \sum_{j \in I} \lambda_j = \mu(I).$$

Let I^+ be the subset of I, where $n(x_i, y_j) = 1$ for every $j \in I^+$. Then (1.4) and (1.5) imply that $\mu(I^+) > \mu(I)/3$. Similarly we get a subset $I^- \subset I$, where $n(x_i, y_j) = -1$ for every $j \in I^-$ and $\mu(I^-) > \mu(I)/3$.

Now we'll describe the process of choice of the rows $i_1, \cdots, i_k$ in the matrix T in order to get the matrix U_k.

Let $\{I_j\}_{j=1}^{2^k}$ be pairwise disjoint subsets of $[1, \cdots, s]$, such that $\bigcup_{j=1}^{2^k} I_j \subset [1, \cdots, s]$ and $\mu(I_j) \geqq 2^{-(k+1)}$. Suppose the vector of signs $\bar{\varepsilon}_{(1)} = (\varepsilon_j^{(1)})_{j=1}^{2^k}$ ($\varepsilon_j^{(1)} = 1$ or -1) is the first row of the matrix U_k. Then we choose the appropriate subsets $I_j^{\varepsilon_j^{(1)}} \equiv I_{j,1}$ ($\mu(I_{j,1}) \geqq \frac{1}{3} \cdot 2^{-(k+1)}$) by the process which was described before. In the choice of every such subset we may "lose" $1/\varepsilon^2 = 36/\mu(I_j)$ rows of the matrix T, hence to complete the choice of the first row of U_k we need $2^k \cdot 36/\mu(I_j) + 1$ rows in T and we can make this choice in each of the remaining $m - 2^k \cdot 36/\mu(I_j)$ rows.

To get another vector of signs (i.e. the next row of the matrix U_k), we repeat the same process for the subsets $\{I_{j,1}\}_{j=1}^{2^k}$. If m is big enough, we can make k steps (choosing in each step a row of U_k) and after it choose 2^k columns in the matrix T, which will give us all the possible vectors of signs in the 2^k columns. On the i-step we "lose" by Lemma I.4, $2^k \cdot 1/\varepsilon_i^2 = 2^k \cdot 36/((\frac{1}{3})^{i-1} \cdot 2^{-(k+1)})$ rows of the matrix T. Since rows which have to be eliminated in step i must not be eliminated in step $j > i$ the estimate on the number of rows of the matrix T is

$$m \geqq k + 2^k \cdot 2^{k+1} \frac{36}{(\frac{1}{3})^k} > 73 \cdot 12^k. \tag{1.6}$$

As mentioned above, the measure of the sets $I_{j,i}$ must exceed $36/n^{1/2}$ on each step. Thus on the last step k we have another restriction on k, namely: $\mu(I_{j,k}) \geqq (\frac{1}{3})^k (\frac{1}{2})^{k+1} \geqq 36/n^{1/2}$. It is easily seen that by taking $m = \frac{1}{4} n^{1/2}$ the last condition is fulfilled when (1.6) holds. From the choice of ε and δ we get $m \geqq \frac{1}{4} n^{1/2}$. Hence, by (1.6) B^* contains a subspace isometric to l_1^k, for every k satisfying $\frac{1}{4} n^{1/2} \geqq 12^k \cdot 73$. Hence asymptotically $(n \to \infty)$ $k \sim (1/2 \ln 12) \ln n$.

2. Now we come to the proof of Theorem 2. Essentially the proof is based on the same approach as the proof of Theorem 1. As in Theorem 1 it is convenient to find a subspace of B^* isomorphic to l_1^k.

We have $d(B^*, l_2^n) \geqq c\sqrt{n}$ and we want to prove the existence of a c'-isomorphic copy of l_1^k in B^* (for some constant c' depending only on c). As above we consider the B^* norm as the conjugate norm of B induced by the inner product (x, y), which is determined by the F. John ellipsoid E.

Now we use Lemma 0.1. As $d(B^*, l_2^n) \geqq c\sqrt{n}$, there is a point $z_1 \in S(B^*)$, such that $\|z_1\|_{l_2^n} \geqq c\sqrt{n}$. Let E^1 be the orthogonal complement of z_1, i.e.

$E^1 = \{z \in B^* \mid (z_1, z) = 0\}$. By Lemma 0.1 $d(E^1, l_2^{n-1}) \geqq \frac{1}{4}c\sqrt{n}$ (if n is big enough), hence there is $z_2 \in E^1$, $\| z_2 \|_{B^*} = 1$ and $\| z_2 \|_{l_1^n} \geqq \frac{1}{4}c\sqrt{n}$. Now consider the subspace $E^2 = \{z \in B^* \mid (z, z_i) = 0,\ i = 1, 2\}$. Again by Lemma 0.1 $d(E^2, l_2^{n-2}) \geqq \frac{1}{4}c\sqrt{n}$, so there is $z_3 \in E^2$, $\| z_3 \|_{B^*} = 1$ and $\| z_3 \|_{l_2^n} \geqq \frac{1}{4}c\sqrt{n}$. Continuing in this manner we can find $m \sim b \log_2 n$ points $\{z_i\}_{i=1}^m$ (b is the constant from Lemma 0.1). So we have proved:

LEMMA 2.1. *Let $d(B^*, l_2^n) \geqq c\sqrt{n}$. There are $m \sim b \log_2 n$ vectors $\{z_i\}_{i=1}^m$, such that $\| z_i \|_{B^*} = 1$, $(z_i, z_j) = 0$ $i \neq j$ and $\| z_i \|_{l_2^n} \geqq \frac{1}{4}c\sqrt{n}$.*

Let $Y = \{y_i\}_{i=1}^s \subset S(B)$ be the set from Proposition I.1. We normalize the measure λ of this set by defining a measure $\mu : \mu(y_i) = \lambda_i / n$. Then (b) and (c) of Proposition I.1 may be written as follows:

$$\text{(b)} \quad \int_Y d\mu(y) = 1 \quad \text{and} \quad \text{(c)} \quad z = \int_Y n(z, y) y \, d\mu(y)$$

for every $z \in B$.

Every $z \in B^*$ (or $z \in B$) can be considered as the function on Y, which obtains the value (z, y_i) for every $y_i \in Y$. Applying (c) to the vectors $\{z_i\}_{i=1}^m$ from Lemma 2.1 with respect to the corresponding $L_2(\mu)$ norm, we get

$$\| z_i \|_{L_2(\mu)}^2 = \int_Y |(z_i, y)|^2 d\mu(y) = \frac{(z_i, z_i)}{n} \geqq \frac{c^2}{16}. \tag{2.1}$$

It is obvious that

$$\| z_j \|_{L_\infty(Y)} \overset{\text{def}}{\equiv} \max_{y_i \in Y} |(z_j, y_i)| \leqq \| z_i \|_{B^*} = 1 \tag{2.2}$$

(since $\| y_i \|_B = 1$ and $\| z \|_{B^*} = \max\{|(z, y)| ; \| y \|_B = 1\}$). Observe that for the proof of Theorem 2 it is enough to find k vectors $\{u_i\}_{i=1}^k \subset B^*$, $\| u_i \|_{B^*} = 1$, such that for some constant $a = a(c) > 0$ and for every k-tuple of scalars $\{\alpha_i\}_{i=1}^k$

$$a \sum_{i=1}^k |\alpha_i| \leqq \left\| \sum_{i=1}^k \alpha_i u_i \right\|_{L_\infty(Y)}. \tag{2.3}$$

In such a case it is obvious that

$$\left\| \sum_{i=1}^k \alpha_i u_i \right\|_{L_\infty(Y)} \leqq \left\| \sum_{i=1}^k \alpha_i u_i \right\|_{B^*} \leqq \sum_{i=1}^k |\alpha_i|$$

and the subspace $E = \operatorname{span}\{u_i\}_{i=1}^k \subset B^*$ is isomorphic to l_1^k with $d(E, l_1^k) \leqq 1/a$. The vectors $\{u_i\}_{i=1}^k$ and the inequality (2.3) will be obtained from Theorem 3, which is rephrased here for our purposes.

Let μ be a probability measure defined on a set T (i.e. $\mu(T)=1$).

THEOREM 3'. *Let $c>0$ and $c_1>0$. For every positive integer k there exists a positive integer m such that for every set of functions $F=\{f_i\}_{i=1}^m \subset L_\infty(T,\mu)$ satisfying $\|f_i\|_{L_\infty} \leqq 1$ $(i=1,\cdots,m)$, $\|f_i\|_{L_2(T,\mu)} \geqq c>0$ and $\|f_i-f_j\|_{L_2(\mu)} \geqq c_1>0$ for $i\neq j$ there is a subset $\{f_{i_1},\cdots,f_{i_{2k}}\}$, such that the differences $\{\phi_j=f_{i_{2j}}-f_{i_{2j-1}}\}_{j=1}^k$ in $L_\infty(\mu)$ are $6/c_1^2$ isomorphic to the standard basis of l_1^k.*

PROOF. For $i\neq j$ it is obvious that $\|f_i-f_j\|_{L_\infty(\mu)} \geqq \|f_i-f_j\|_{L_2(\mu)} \geqq c_1>0$ $(i,j=1,\cdots,m)$. Hence by the theorem of Brunel and Sucheston ([1]), given a positive integer k there is a positive integer M, such that for every $m\geqq M$ there exists a subset $A=\{f_{i_1},\cdots,f_{i_{2k}}\}\subset F$, such that the differences $\{\phi_j=f_{i_{2j}}-f_{i_{2j-1}}\}_{j=1}^k$ form an unconditional basis with constant less than 3 (in the $L_\infty(\mu)$ norm). Observe that

$$\|\phi_j\|_{L_1(\mu)}=\int_T|\phi_j(t)|\,d\mu(t)\geqq\frac{\|\phi_j\|_{L_\infty(\mu)}}{2}\int|\phi_j(t)|\,d\mu(t)$$

$$\geqq\frac{1}{2}\int_T|\phi(t)|^2\,d\mu(t)\geqq c_1^2/2.$$

Now we make use of the fact that the embedding operator $i:L_\infty(\mu)\to L_1(\mu)$ is absolutely summing[†] (see [15]). Hence

$$\frac{c_1^2}{2}\sum_{j=1}^k|a_j|\leqq\sum_{j=1}^k|\alpha_j|\;\|\phi_j\|_{L_1(\mu)}\leqq\sup_{\pm}\left\|\sum_{j=1}^k\pm\alpha_j\phi_j\right\|_{L_\infty(\mu)}$$

$$\leqq 3\left\|\sum_{j=1}^k\alpha_j\phi_j\right\|_{L_\infty(\mu)}\leqq 3\sum_{j=1}^k|\alpha_j|.$$

So, $\{\phi_0\}_{j=1}^k$ is $6/c_1^2$ equivalent to the standard basis of l_1^k.

REMARK. In our case $F=\{z_i\}_{i=1}^m$ where by Lemma 2.1 $(z_i,z_j)=0$ $(i\neq j)$. Hence $\|z_i-z_j\|_{L_2(\mu)}\geqq\sqrt{2}c=c_1$. In this case (i.e. when the functions $\{z_i\}_{i=1}^m$ are orthogonal) it is unnecessary to consider the differences $\phi_j=f_{i_{2j}}-f_{i_{2j-1}}$ $(j=1,\cdots,k)$ and we can choose a subset $A\subset F$ equivalent to the standard basis of l_1^k. In Appendix 2 we present the first version of the proof of Theorem 3 which does not use the result of Brunel and Sucheston and gives the estimate $k\sim\log\log m$.

† An operator $T:X\to Y$ is absolutely summing (with norm $\pi_1(T)\leqq c$), if for every n and every n-tuple $\{x_1,\cdots,x_n\}\subset X$:

$$\sum_{i=1}^n\|Tx_i\|\leqq c\sup_{\|x^*\|\leqq 1}\sum_{i=1}^n|x^*(x_i)|=c\max_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^n\varepsilon_i x_i\right\|.$$

Appendix 1

Proof of Lemma 0.1 (Maurey). We use an equivalent definition of $d(E, l_2^n)$ (see [2], [13]):

Let $\{e_i\}_{i=1}^m$ be an orthogonal basis in l_2^m and let $\{\eta_i\}_{i=1}^m$ be the standard basis of l_1^m. Let $\{\alpha_i\}_{i=1}^m$ be an n-tuple of scalars satisfying $\Sigma_{i=1}^m \alpha_i^2 \leqq 1$. Suppose α, T are mappings $l_2^m \xrightarrow{\alpha} l_1^m \xrightarrow{T} E$ such that $\alpha(e_i) = \alpha_i \eta_i$ $(i = 1, \cdots, m)$ and $\| T \| \leqq 1$. Then $d(E, l_2^n) = \sup\{\pi_2(T\alpha)\}$, where the supremum is taken over all m positive integers and all operators α and T such that $\Sigma_{i=1}^m \alpha_i^2 \leqq 1$ and $\| T \| \leqq 1$.

Here $\pi_2(T\alpha)$ is the 2-summing norm of the operator $T\alpha$ (see [15]).†

Let $m > 0$, α and T be such that $l_2^m \xrightarrow{\alpha} l_1^m \xrightarrow{T} B$, $\| T \| \leqq 1$, $\Sigma_{i=1}^m \alpha_i^2 \leqq 1$ and $\pi_2(T\alpha) \simeq d(B, l_2^{n+k})$. Assume that we can find an operator $U : l_1^m \to B$ such that $\| U \| \leqq 2$, $(T - U) : l_1^m \to E \subset B$ (i.e. $\mathrm{Im}(T - U) \subset E$) and $p = \dim U l_1^m \leqq 2^{bk}$. Then $\| T - U \| \leqq 3$ and since $d(E, l_2^n) = d$ it follows that

(i) $$\pi_2((T - U)\alpha) \leqq 3d.$$

On the other hand $\| U \| \leqq 2$ and by F. John ([11]) $d(U l_1^m, l_2^p) \leqq \sqrt{p}$. Hence

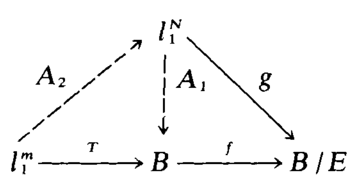
(ii) $$\pi_2(U\alpha) \leqq 2\sqrt{p} \leqq 2 \cdot 2^{\frac{1}{2}b \cdot k}.$$

Inequalities (i) and (ii) imply

$$d(B, l_2^{n+k}) \simeq \pi_2(T\alpha) \leqq \pi_2((T - U)\alpha) + \pi_2(U\alpha) \leqq 3d + 2 \cdot 2^{\frac{1}{2}bk}.$$

So, in order to complete the proof we have to find the operator U.

Let f be the natural factorization mapping $f : B \to B/E$. It is well known that there exists a positive integer $N = 2^{bk}$, $b > 0$ such that there is a mapping g from l_1^N onto B/E, $g : l_1^N \to B/E$, satisfying $\| g \| \leqq 2$ (N is taken as the number of points of an ε_0-net (for some $\varepsilon_0 > 0$) on the surface of the unit sphere of B/E). Hence, we have the diagram



By the lifting property of l_1 ([15]) there are operators $A_1 : l_1^N \to B$ and $A_2 : l_1^m \to l_1^N$ such that $fA_1 = g$, $\| A_1 \| = \| g \|$ and $gA_2 = fT$, $\| A_2 \| = \| f \cdot T \| \leqq \| f \| \; \| T \| \leqq 1$.

† Let $T : X \to Y$. Then $\pi_2(T)$ is the smallest constant c, such that for every n and $\{x_i\}_{i=1}^n \subset X$ the inequality $\Sigma_{i=1}^n \| Tx_i \|^2 \leqq c^2 \sup_{\|x^*\| \leqq 1} \Sigma_{i=1}^n | x^*(x_i) |^2$ holds.

Let $U = A_1A_2$. Then $fT = fU$, since the diagram is commutative, which means that $f(T-U)=0$ and, hence, $(T-U): l_1 \to E$. The other conditions on U hold too, since $\dim Ul_1^m \leqq \dim l_1^N = N$ and $\| U \| \leqq \| A_1 \| \, \| A_2 \| \leqq \| g \| \leqq 2$.

Appendix 2

Proof of Theorem 3. For the sake of simplicitly we will prove the theorem for the case of $T=[0,1]$ and μ-being the ordinary Lebesque measure on the interval $[0,1]$. Let $a>0$. Assume that there are k functions $\{\phi_s\}_{s=1}^k \subset \{f_i\}_{i=1}^m$ such that for every combination of signs $\bar{\varepsilon}_j = (\varepsilon_j(i) = \pm 1)_{i=1}^k$ there is a set J_j, $\mu(J_j)>0$, where $\{\varepsilon_j(i)\cdot\phi_i(t)>a,\ \forall t\in J_j\}_{i=1}^k$. Then for every k-tuple of scalars $\{a_i\}_{i=1}^k$

$$a\sum_{i=1}^k |a_i| \leqq \left\| \sum_{i=1}^k a_i\phi_i \right\|_{L_\infty[0,1]} \leqq A\sum_{i=1}^k |a_i|$$

and $\operatorname{span}\{\varphi_i\}_{i=1}^k \overset{\text{def}}{\equiv} [\phi_i]_{i=1}^k$ is A/a-isomorphic to l_1^k. Therefore we have to find k functions and 2^k subsets $\{J_j\}_{j=1}^{2^k}$ of $[0,1]$ satisfying these conditions.

Lemma A.1. *Let $\{f_i\}_{i=1}^m$ and $A \geqq 1$ be as defined in Theorem 3. Then given $0<\alpha<1$ there are m subsets $\{G_i \subset [0,1])\}_{i=1}^m$, such that $G_i = \{t\in[0,1] \mid |f_i(t)| \geqq \alpha\}$ and $\mu(G_i) \geqq \beta$ for every i, where $\beta = (1-\alpha^2)/(A^2-\alpha^2)$.*

The proof is obvious:

$$A^2\mu(G_i) \geqq \int_{G_i} |f_i(t)|^2 d\mu = \int_T |f_i(t)|^2 d\mu - \int_{T\setminus G_i} |f_i(t)|^2 d\mu$$
$$\geqq 1-\alpha^2(1-\mu(G_i)).$$

Therefore $\mu(G_i) \geqq (1-\alpha^2)/(A^2-\alpha^2)$.

Lemma A.2. *Let $\{G_i\}_{i=1}^m$ and β be as defined in Lemma A.1 and let $1>\mu>1/m$ be such that $1/\mu$ is an integer. Then there is an interval $G\subset T=[0,1]$ of measure μ, such that $\mu(G\cap G_i)/\mu(G) \geqq \beta$ for at least $\mu\cdot m$ sets G_i.*

Proof. Split T into $1/\mu = p$ pairwise disjoint intervals $\{F_j\}_{j=1}^p$ of measure μ each. By Lemma A.1, $\mu(G_i)\geqq\beta$ for every i, therefore given $1\leqq i_0 \leqq m$ there is $1 \leqq j_0 \leqq p$ such that $\mu(F_{j_0}\cap G_{i_0})/\mu(F_{j_0}) \geqq \beta(*)$. On the other hand, if no F_{j_0} satisfies the lemma, then $(*)$ is satisfied by less than $m\mu\cdot p = m$ sets G_i, which is a contradiction.

Let $0<\alpha<1$ be a scalar and let $\beta=(1-\alpha^2)/(A^2-\alpha^2)$. Take $\beta/2^{k+1} \leqq \mu \leqq \beta/2^k$ such that $1/\mu$ is an integer. Denote by $F=\{f_i\}_{i=1}^m$. Let the sets G_i be as defined in Lemma A.1. Then by Lemma A.2 there is an interval $I_1\subset T$ and there is a subset $F^1 \subset F$ such that $\mu(I_1) = \mu \geqq \beta/2^{k+1}$ and $\mu(G_i \cap I_1)/\mu(I_1) \geqq \beta$ for every i such that $f_i \in F^1$, where $|F^1| \geqq \mu \cdot m$.

In the complement of the set I_1 we get $\mu(G_i \backslash I_1) \geqq \beta - \beta/2^{k+1}$ and the density† of the set G_i in the set $I \backslash I_1$ is:

$$\frac{\mu(G_i \backslash I_1)}{\mu(I \backslash I_1)} \geqq \beta\left(1 - \frac{1}{2^{k+1}}\right).$$

By Lemma A.2 we find an interval $I_2 \subset I \backslash I_1$, $\mu(I_2) = \mu$, and a subset $F^2 \subset F^1$ ($|F^2| \geqq \mu^2 m$) such that for every i, where $f_i \in F^2$:

$$\frac{\mu(G_i \cap I_2)}{\mu(I_2)} \geqq \beta\left(1 - \frac{1}{2^{k+1}}\right).$$

After 2^k steps we get 2^k pairwise disjoint intervals of $T: I_1, \cdots, I_{2^k}$ and a set $F_1 \subset \bigcap_{i=1}^{2^k} F^i$ such that $\mu(I_j) = \mu$ $(\beta/2^{k+1} \leqq \mu \leqq \beta/2^k)$ for $1 \leqq j \leqq 2^k$ and for every i such that $f_i \in F_1$

$$\frac{\mu(I_j \cap G_i)}{\mu(I_j)} \geqq \frac{\beta}{2} \overset{\text{def}}{\equiv} \beta_1 \quad (1 \leqq j \leqq 2^k).$$

At the same time we have

$$|F_1| \geqq \mu^{2^k} |F| \geqq \left(\frac{\beta}{2^{k+1}}\right)^{2^k} \cdot m. \tag{A.1}$$

Now we describe the choice of the first function $\phi_1 \in F_1$. Similarly to the proof of Theorem 1 let U_k denote the $k \times 2^k$-dimensional matrix having all the possible 2^k vectors of signs as its columns.

Step A. Let $\bar{\varepsilon}_1 = (\varepsilon_1(j) = \pm 1)_{j=1}^{2^k}$ be the first row of the matrix U_k. Define for every i

$$G_{i,1} = I_1 \cap G_i \quad \text{and} \quad \bar{G}_{i,1} = \left\{ t \in I_1 \,\middle|\, |f_i(t)| \geqq \frac{\beta_1 \alpha}{6} \right\}$$

(evidently $\overline{G_{i,1}} \supset G_{i,1}$). For every i such that $f_i \in F_1$ we know that $\mu(G_i \cap I_1) \geqq \beta_1 \mu(I_1)$.

Hence

$$\int_{\bar{G}_{i,1}} |f_i(t)|\, dt > \int_{G_{i,1}} |f_i(t)|\, dt \geqq \alpha \beta_1 \mu(I_1). \tag{A.2}$$

Let us consider a functional in $L_2[0,1]$ determined by the function

† The density of a set G in a set I is defined as $\mu(G \cap I)/\mu(I)$.

$$\psi(t)=\begin{cases}1 & t\in I_1\\ 0 & t\notin I_1\end{cases}$$

and apply Lemma 1.3 to this functional and to the orthonormal system $\{f_i\}_1^m$. Since $\|\psi\|_{L_2}=\sqrt{\mu(I_1)}$ we obtain

$$\begin{aligned}\left|\int_{\overline{G}_{i,1}} f_i(t)dt\right| &\leqq \left|\int_{I_1\setminus\overline{G}_{i,1}} f_i(t)dt\right| + \left|\int_{I_1} f_i(t)dt\right| \\ &\leqq \frac{1}{6}\beta_1\alpha\mu(I_1\setminus\overline{G}_{i,1})+\delta_1\sqrt{\mu(I_1)}\end{aligned} \tag{A.3}$$

except for at most $1/\delta_1^2$ functions $\{f_i\}$ from F_1. By taking $\delta_1=\frac{1}{6}\beta_1\alpha\sqrt{\mu(I_1)}$ we get

$$\left|\int_{\overline{G}_{i,1}} f_i(t)dt\right| \leqq \frac{1}{3}\beta_1\alpha\mu(I_1) \tag{A.4}$$

for every $f_i\in F_1$ except for at most $1/\delta_1^2$ of them. The inequalities (A.2), (A.4) and $\|f_i\|_{L_\infty}\leqq A$ imply that *there are subsets* $J^+_{1,i,1}$ *and* $J^-_{1,i,1}$ *of* $\overline{G}_{i,1}$ *(for every* $f_i\in F_1$ *except* $1/\delta_1^2$ *of them) with measures not less than* $\frac{1}{3}\cdot(\beta_1\alpha/A)\mu(I_1)$ *such that*

$$\operatorname{sign} f_i(t)=\begin{cases}+1 & t\in J^+_{1,i,1}\\ -1 & t\in J^-_{1,i,1}\end{cases}.$$

We denote $J_{1,i,1}=J^+_{1,i,1}$ if $\varepsilon_1(1)=+1$ and $J_{1,i,1}=J^-_{1,i,1}$ if $\varepsilon_1(1)=-1$. In the exceptional case when $\mu(J_{1,i,1})$ exceeds $\frac{1}{3}\mu(I_1)$ we will prefer, for purely technical reasons, to denote by $J_{1,i,1}$ a measurable subset $J_{1,i,1}$ of the measure $=\frac{1}{3}\mu(I_1)$.

The immediate consequence of Step A is

Step B. For each $f_i\in F_1$ except for at most $2^k\cdot 1/\delta_1^2$ of them we can find subsets $J_{1,i,j}\subset\bar{G}_{i,j}$ $(1\leqq j\leqq 2^k)$ satisfying:

(a) $(\frac{1}{3}\mu\geqq)\,\mu(J_{1,i,j})\geqq\frac{1}{3}\frac{\beta_1\alpha}{A}\mu\ \left(\geqq\frac{1}{3}\frac{\beta_1\alpha}{A}\cdot\frac{\beta_1}{2^k}\right)$,

(b) $|f_i(t)|\geqq\frac{1}{6}\beta_1\alpha,\quad \forall t\in J_{1,i,j}\ (j=1,\cdots,2^k)$,

(c) $\operatorname{sign} f_i(t)=\varepsilon_1(j),\ t\in J_{1,i,j}$.

We denote this set of functions by $F_{1,0}\subset F_1$, $|F_{1,0}|\geqq|F_1|-2^k/\delta_1^2$. Note that $\delta_1=\frac{1}{6}\beta_1\alpha\sqrt{\min_j\mu(I_j)}$, however, at this stage $\mu(I_j)\equiv\mu$.

Step C. Assume that there is a function $f_{i_0}\in F_{1,0}$ such that for every set $J_{1,i_0,j}$ $(1\leqq j\leqq 2^k)$ there are no more than $\gamma|F_1|$ (we will use the number $\gamma=1/2^{k+1}$) functions from F_1 for which the density of the set G_i in $J_{1,i_0,j}$ is less than $\sigma\beta_1$ (the choice of the number σ is determined by the equation $\sigma^k=\frac{1}{2}$ and will be justified at the end of the proof).

In this case there is a subset $F_2 \subset F_1$, $|F_2| \geqq (1-\gamma \cdot 2^k)\cdot |F_1|$ such that for each $f_i \in F_2$

$$\mu(G_i \cap J_{1,i_0,j})/\mu(J_{1,i_0,j}) \geqq \sigma\beta_1 \quad (1 \leqq j \leqq 2^k).$$

Then f_{i_0} is taken for ϕ_1 and we return to Step A with the sets $J_{1,i_0,j} \overset{\text{def}}{\equiv} I_j^{(2)}$ instead of I_j, with the set F_2 instead of F_1, and we continue the process to find ϕ_2 corresponding to the second row of signs $\bar{\varepsilon}_2$ of the matrix U_k.

Step D. If the assumption of Step C is not true then there is a subset $J_{1,i_0,j_0} \subset \bar{G}_{i_0,j_0}$ and a set $H_1 \subset F_1$, $|H_1| \geqq \gamma |F_1|$, such that for each $f_i \in H_1$,

$$\mu(G_i \cap J_{1,i_0,j_0})/\mu(J_{1,i_0,j_0}) < \sigma\beta_1.$$

Consider $I_{2,j_0} = I_{j_0} \setminus J_{1,i_0,j_0}$. It is clear (see Step B (a)) that

$$\frac{2}{3}\mu(I_{j_0}) \leqq \mu(I_{2,j_0}) \leqq \left(1 - \frac{1}{3}\frac{\beta_1\alpha}{A}\right)\mu(I_{j_0})$$

and for each $f_i \in H_1$ the density of the set G_i in I_{2,j_0} is increased to

$$\beta_2 \equiv \frac{\mu(G_i \cap I_{2,j_0})}{\mu(I_{2,j_0})} \geqq \frac{\beta_1\mu(I_{j_0}) - \sigma\beta_1\mu(J_{1,i_0,j_0})}{\mu(I_{j_0}) - \mu(J_{1,i_0,j_0})} \geqq \frac{\beta_1 - \theta\sigma\beta_1}{1-\theta} = \beta_1\left[1 + \frac{1-\sigma}{1-\theta}\theta\right],$$

where $\theta = \frac{1}{3}\beta_1\alpha / A$. This discussion proves the first part of the following lemma.

LEMMA A.3. *Let* $0 < \theta < 1$, $0 < \sigma < 1$ *and* $\{J_r \subset I_r \subset T\}_{r=0}^p$ *be a system of measurable sets such that* $J_{r+1} \subset I_r \setminus J_r = I_{r+1}$ $(r = 0, \cdots, p-1)$ *and* $\mu(J_r) \geqq \theta\mu(I_r)$ *for every* $r = 0, \cdots, p$. *Let* $G \subset T$ *and for every* $r = 0, \cdots, p$ *we denote* $\mu(G \cap I_r)/\mu(I_r) = \beta_r$. *Assume that* $\mu(G \cap J_r)/\mu(J_r) \leqq \sigma\beta_r$ $(\forall r = 0, \cdots, p-1)$. *Then the density* β_{r+1} *of the set* G *in* I_{r+1} *is increased and*

$$\beta_{r+1} \geqq \beta_r\left(1 + \frac{1-\sigma}{1-\theta}\theta\right).$$

Besides that, the number of the indicated sets p *does not exceed*

$$p \leqq \frac{\log \dfrac{1}{\beta_0}}{\log\left(1 + \dfrac{1-\sigma}{1-\theta}\theta\right)} \simeq \left(\log\frac{1}{\beta_0}\right)\frac{1-\theta}{\theta(1-\sigma)} \quad \textit{for small } \theta.$$

PROOF. We also have to prove the estimate on p which follows trivially from the condition $\beta_p \leqq 1$ and the previous estimate

$$\beta_p \geqq \left(1+\frac{1-\sigma}{1-\theta}\theta\right)^p \beta_0.$$

REMARK. If $\sigma^k = \frac{1}{2}$ we have $k \log(1-(1-\sigma)) = -\log 2$ and $k(1-\sigma) \simeq \log 2$. Therefore, substituting β_0 by the number β_s and θ by the number $\frac{1}{3}\beta_s\alpha/A$ in the inequality for p, we obtain

$$\text{(A.5)} \qquad p \lesssim \frac{\log \frac{1}{\beta_s}}{\log 2} \frac{k}{\theta} = \frac{\log \frac{1}{\beta_s}}{\log 2} \frac{3A}{\beta_s \alpha} k.$$

Thus, we are finishing Step D with a new set of functions H_1 and new sets $\{I_{2,j}\}_{j=1}^{2^k}$ (which at the first stage, all with the exception of $I_{2,j_0} = I_{j_0} \setminus J_{1,i_0,j_0}$ coincide with I_j).

Then we check, for the new set of functions H_2 and the new sets $\{I_{2,j}\}_{j=1}^{2^k}$ whether the conditions of Step C are satisfied. If they are not, then we return to the conditions of Step D. However, by Lemma A.3, Step D cannot be repeated more than $2^k \cdot p$ times running since for every j Step D by Lemma A.3 takes place not more than p times and the amount of the numbers j equals 2^k. Thus, after not more than $2^k \cdot p$ repetitions of Step D the conditions of Step C must obtain for some subset $F_2' \subset F_1$ instead of F_1 where

$$|F_2'| \geqq \gamma^{2^k \cdot p} |F_1|$$

and for some sets $\{I_j^{(2)} \subset I_j \subset [0,1]\}_{j=1}^{2^k}$ instead of $\{I_j\}_{j=1}^{2^k}$. At the same time the measure of the sets $I_j^{(2)}$ ($j = 1, \cdots, 2^k$) is decreased not more than $(\frac{2}{3})^p$ times as a result of the application of Step D and the execution of Step C leads to a possible additional decrease of the measure of the set in question by the inequality of Step B (a):

$$\text{(A.6)} \qquad \mu(I_j^{(2)}) \geqq \frac{1}{3}\frac{\beta_1\alpha}{A}\left(\frac{2}{3}\right)^p \mu(I_j) \geqq \left(\frac{2}{3}\right)^p \frac{\beta_1}{2^k} \cdot \frac{1}{3}\frac{\beta_1\alpha}{A}.$$

Besides, the execution of Step C means that we have chosen the function $\phi_1 = f_{i_0} \in F$ and the sets $\{I_{1,i_0,j}\}_{j=1}^{2^k}$ (as noted in Step C) and now we go on to the execution of Step A (in order to find a function ϕ_2) with the density β_2 of the sets G_i (for $f_i \in F_2$) in $I_j^{(2)}$ where

$$\beta_2 \geqq \sigma\beta_1.$$

At the same time the set of functions F_2 which we can use at Step A has decreased (see Step C) but

$$|F_2| \geqq (1-2^k\gamma)|F_2'| \geqq (1-2^k\gamma)\gamma^{2^k \cdot p}|F_1|.$$

To construct the k functions $\phi_1, \cdots, \phi_k$ (corresponding to the k rows of the matrix of signs U_k) we have to go back to Step A k times. Thus the set of functions F_k is estimated at the last stage as follows:

$$|F_k| \geqq (1-2^k\gamma)^k (\gamma^{2^k \cdot p})^k \cdot |F_1|. \tag{A.7}$$

The density

$$\beta_k \geqq \sigma^k \beta_1 = \frac{1}{2}\beta_1$$

and the measure of the sets $\{I_j^{(k)} \subset [0,1]\}_{j=1}^{2^k}$ is then estimated by the application of the inequality (A.6) k times:

$$\mu(I_j^{(k)}) \geqq \prod_{s=1}^{k} \left(\frac{1}{3}\frac{\beta_s \alpha}{A}\right) \cdot \left(\frac{2}{3}\right)^{p \cdot k} \cdot \frac{\beta_1}{2^k} \geqq \left(\frac{1}{6}\frac{\beta_1\alpha}{A}\right)^k \left(\frac{2}{3}\right)^{k \cdot p} \frac{\beta_1}{2^k}. \tag{A.8}$$

The execution of Step B is possible then only in the case if $|F_k| > 2^k/\delta_k^2$ where δ_k depends on the measure $\min_j \mu(I_j^{(k)})$ and by (A.8)

$$\delta_k^2 = \left(\frac{1}{6}\beta_k\alpha \cdot \min_j \sqrt{\mu(I_j^{(k)})}\right)^2 \geqq \left(\frac{1}{12}\cdot\beta_1\alpha\right)^2 \left(\frac{1}{6}\frac{\beta_1\alpha}{A}\right)^k \left(\frac{2}{3}\right)^{k\cdot p} \frac{\beta_1}{2^k}.$$

Therefore, it follows from (A.7) that the fulfillment of the inequality

$$(1-2^k\gamma)^k \gamma^{2^k \cdot k \cdot p} \cdot |F_1| \geqq 2^k \left(\frac{12}{\beta_1\alpha}\right)^2 \left(\frac{6A}{\beta_1\alpha}\right)^k \left(\frac{3}{2}\right)^{k\cdot p} \cdot \frac{2^k}{\beta_1} \tag{A.9}$$

ensures the possibility to execute Step B and thus choose the last function $\phi_k \in F_k$. Thus, we have constructed the sets $\{I_j^{(k)}\}_{j=1}^{2^k}$, $\mu(I_j^{(k)}) > 0$ $(j = 1, \cdots, 2^k)$ such that

$$|\phi_i(t)| \geqq \frac{\beta_k\alpha}{6} \geqq \frac{1}{12}\beta_1\alpha \overset{\text{def}}{=} a \quad (\forall t \in I_j^{(k)} \text{ on all } j = 1, \cdots, 2^k),$$

and for any combination of k signs $\bar{\varepsilon} = (\varepsilon_1 = \pm 1)_{i=1}^k$ there is a set $I_j^{(k)} = I(\bar{\varepsilon}) \subset [0,1]$ such that for $t \in I(\bar{\varepsilon})$

$$\operatorname{sign} \phi_i(t) = \varepsilon_i.$$

As noted at the beginning of the proof this means that the subspace $[\phi_i]_{i=1}^k \subset L_\infty[0,1]$ is isomorphic to l_1^k and the constant of the isomorphism of $\{\phi_i\}_1^k$ and the natural basis l_1^k does not exceed

$$d = A/a \leqq 12A/\alpha\beta_1. \tag{A.10}$$

Let us now take $\gamma = 1/2^{k+1}$ and $\alpha = \frac{1}{2}$ in Lemma A.1 and recall that $\beta_1 = \beta/2 = \frac{1}{2}(1-\alpha^2)/(A^2-\alpha^2) \simeq b/A^2$ (we are no longer interested in the values of the absolute constants and denote all of them by b). Besides, from (A.5) we obtain (by force of $\beta_s \geqq \sigma^s\beta_1 \geqq \frac{1}{2}\beta_1$)

$$p \lesssim bA^3 \log A \cdot k.$$

Combining this inequality with (A.9) and the estimate $|F_1|$ from (A.1), we obtain that it is possible to choose $k \sim \ln \ln m$ and by (A.10) the distance $d([\phi_i]_1^k, l_1^k) \leqq d \leqq bA^3$.

REMARK. Applying the last estimates to Theorem 2 we obtain that there is a k-dimensional ($k \sim \log\log\log n$) subspace which is $b(1/c^3)$-isomorphic to l_1^k. (Here $A = 4/c$ and $m \simeq \log n$, see Lemma 2.1 and (2.1).) It is well known that if $\{x_i\}_{i=1}^{k^2}$ is a normalized basic sequence, which is κ^2-equivalent to the standard basis of $l_1^{k^2}$ then there is a normalized block basic sequence $\{z_i\}_{i=1}^k \subset [x_i]_{i=1}^{k^2}$, which is κ-equivalent to the standard basis of l_1^k (see [10] and [7]).

By repetitiously applying this last argument we obtain that if in Theorem 2 the constant $c = c(n) = (\log\log\log n)^{-o(1)}$ then for every $\varepsilon > 0$ we can still get k-dimensional subspaces of B ε-isometric to l_1^k such that $k \nearrow \infty$ when $n \nearrow \infty$. This proves the Remark after Theorem 2 in the introduction.

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